UNIT = II

Closed sets:

Definition: Let (M, d) be a metric space. Let $A \subseteq M$. Then A is said to be closed in M if the complement of A is open in M.

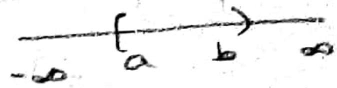
Ex 1: In R with usual metric any closed interval $[a,b]$ is closed set.

Proof: $[a,b]^c = R - [a,b] = (-\infty, a) \cup (b, \infty)$

Also $(-\infty, a)$ & (b,∞) are open in R.

i.e) $[a,b]^c$ is open in R.

$\therefore [a,b]$ is closed in R.



Ex 2: In R with usual metric $[a,b)$ is neither closed nor open.

Proof: $[a,b)$ is not open in R. $\because$ a is not an interior point of $[a,b)$.

Now, $[a,b)^c = R - [a,b)$
$= (-\infty, a) \cup [b, +\infty)$ and this set is not open. $\because$ b is not an interior point.

$\therefore [a,b)$ is not closed in R.

Hence $[a,b)$ is neither open nor closed in R.

Ex 3: In R with usual metric $(a,b]$ is neither closed nor open.

Proof: $(a,b]$ is not open in R.

$\because$ b is not an interior point of $(a,b]$

Now $(a,b]^c = R - (a,b]$
$= (-\infty, a] \cup (b,\infty)$ & this set is not open since a is not an interior point.

$\therefore (a,b]$ is not closed in R.

Hence $(a,b]$ is neither open nor closed in R.

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Ex 4: Z is closed.

Proof:

$$Z^c = \bigcup_{n=-\infty}^{\infty} (n, n+1)$$

The open interval $(n, n+1)$ is open and union of open sets is open.

Z^c is open. Hence Z is closed.

Ex 5: $\mathbb{Q}$ is not closed in $\mathbb{R}$.

Proof:

$\mathbb{Q}^c$ = the set of irrationals which is not open in $\mathbb{R}$.

$\therefore \mathbb{Q}$ is not closed in $\mathbb{R}$.

Ex 6: The set of irrational number is not closed in $\mathbb{R}$. (H.W)

Ex 7: In $\mathbb{R}$ with usual metric every sigleton set is closed.

Proof: Let $a \in \mathbb{R}$.

Then $\{a\}^c = \mathbb{R} - \{a\} = (-\infty, a) \cup (a, \infty)$

$\because (-\infty, a)$ and (a, ∞) are both open sets

$(-\infty, a) \cup (a, \infty)$ is open.

$\therefore \{a\}^c$ is open $\mathbb{R}$.

$\therefore \{a\}$ is closed in $\mathbb{R}$.

Ex 8 Every subset of a discrete metric space is ~~open~~ A^c ~~is open~~. Closed.
(Q)

Proof: Let (M, d) be a discrete metric space.

Let $A \subseteq M$.

$\therefore$ Every subset of a discrete metric space is open A^c is open.

$\therefore A$ is closed.

Definition: Let (M, d) be a metric space. Let $a \in M$. Let r be any positive real number. Then the <u>closed ball</u> or the <u>closed sphere</u> with centre a and radius r, denoted by $B_d[a, r]$ is defined by $B_d[a, r] = \{x \in M / d(a, x) \leq r\}$. When the metric d under consideration is clear we write $B[a, r]$ instead of $B_d[a, r]$

Ex 1: In R with usual metric

$$B[a,r] = \{x / d(a,x) < r\}$$
$$= \{x / |x-a| < r\}$$
$$= \{x / -r \leq (x-a) \leq r\}$$
$$= [a-r, a+r]$$

Ex 2: In R^2 with usual metric let $a = (a_1, a_2) \in R^2$

$$\Rightarrow B[a,r] = \{(x,y) \in R^2 / d((a_1,a_2),(x,y)) \leq r\}$$
$$= \{(x,y) \in R^2 / (x-a_1)^2 + (y-a_2)^2 \leq r^2\}.$$

$\therefore B[a,r]$ is the set of all points which lie within and on the circumference of the circle with centre a and radius r.

Theorem 2.8: In any metric space every closed ball is a closed set.

Proof: Let (M,d) be a metric space.

Let $B[a,r]$ be a closed ball in M.

Case i): Suppose $B[a,r]^c = \phi$

$\therefore B[a,r]^c$ is open and hence $B[a,r]$ is closed.

Case ii): Suppose $B[a,r]^c \neq \phi$

Let $x \in B[a,r]^c$

$\therefore x \notin B[a,r]$

$\therefore d(a,x) > r$

$d(a,x) - r > 0$.

Let $r_1 = d(a,x) - r$

To prove: $B(x,r_1) \subseteq B[a,r]^c$

Let $y \in B(x,r_1)$

$\Rightarrow d(x,y) < r_1 = d(a,x) - r$.

$\therefore d(a,x) > d(x,y) + r \longrightarrow (1)$

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Now, $d(a,x) \le d(a,y) + d(y,x)$

$\therefore d(a,y) \ge d(a,x) - d(y,x)$

$> d(x,y) + r - d(y,x)$ (by ①)

$= r$

Thus $d(a,y) > r$.

$\therefore y \notin B[a,r]$.

Hence $y \in B[a,r]^c$.

$\therefore B(x,r_1) \subseteq B[a,r]^c$.

$\therefore B[a,r]^c$ is open in M.

$\therefore B[a,r]$ is closed in M.

Theorem 2.9: In any metric space M, i) ϕ is closed, ii) M is closed

Proof: Since $M^c = \phi$ is open. M is closed

Similarly $\phi^c = M$ is open, ϕ is closed.

Note: In any metric space M, ϕ and M are both open and closed.

Theorem 2.10: In any metric space arbitrary intersection of closed sets is closed.

Proof: Let (M,d) be a metric space.

Let $\{A_i / i \in I\}$ be a collection of closed sets

To prove: $\bigcap_{i \in I} A_i$ is closed.

we have $\left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^c$ (by De Morgan's law)

Since A_i is closed A_i^c is open.

$\therefore \bigcup_{i \in I} A_i^c$ is open

$\therefore \left(\bigcap_{i \in I} A_i\right)^c$ is open,

$\therefore \bigcap_{i \in I} A_i$ is closed.

Theorem 2.11: In any metric space, the union of a finite number of closed sets is closed.

Proof: Let (M, d) be a metric space.

Let $A_1, A_2, \ldots, A_n$ be closed sets in M.

By De-Morgan's law $(A_1 \cup A_2 \cup \ldots \cup A_n)^c$

$$= A_1^c \cap A_2^c \cap \ldots \cap A_n^c.$$

$\because$ each A_i is closed A_i^c is open.

Hence $A_1^c \cap A_2^c \cap \ldots \cap A_n^c$ is open.

$\therefore (A_1 \cup A_2 \ldots \cup A_n)^c$ is open.

Hence $A_1 \cup A_2 \cup \ldots \cup A_n$ is closed.

Note: The union of an infinite collection of closed sets need not be closed. For example, consider R with usual metric.

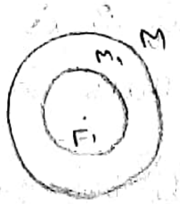
Let $A_n = [\frac{1}{n}, 1]$ where $n = 1, 2, 3, \ldots$

$$\Rightarrow \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} [\tfrac{1}{n}, 1] = \{1\} \cup [\tfrac{1}{2}, 1] \cup [\tfrac{1}{3}, 1] \cup \ldots$$

$$= (0, 1] \text{ which is not closed in } R.$$

$\therefore \bigcup_{n=1}^{\infty} A_n$ is not closed.

Theorem 2.12: Let M be a metric space and M_1 be a subspace of M. Let $F_1 \subseteq M_1$. Then F_1 is closed in M_1 iff there exists a set F which is closed in M such that $F_1 = F \cap M_1$.



Proof: Let F_1 be closed in M_1.

$\therefore M_1 - F_1$ is open in M_1.

$\therefore M_1 - F_1 = A \cap M_1$ where A is open in M.

Now, $F_1 = M_1 - (A \cap M_1)$

$= M_1 - A$ $\quad [\because$ formula $A - B = A \cap B^c]$

$= A^c \cap M_1$

Since A is open in M, A^c is closed in M.

Closure:

Let (M, d) be a metric space. Let $A \subseteq M$. consider the collection of all closed sets which contain A. This collection is non empty since at least M is a member of this collection.

Definition: Let A be a subset of metric space (M, d). The closure of A, denoted by $\bar{A}$ is defined to be the intersection of all closed sets which contain A.

Thus $\bar{A} = \cap \{B / B$ is closed in M and $A \subseteq B\}$.

Note: Since intersection of any collection of closed sets is closed $\bar{A}$ is a closed set. Further $\bar{A} \supseteq A$. Also if B is any closed set containing A then $\bar{A} \subseteq B$. Thus $\bar{A}$ is the smallest closed set containing A.

Theorem 2.13: A is closed iff $A = \bar{A}$

Proof:

Suppose $A = \bar{A}$

$\therefore \bar{A}$ is closed A is closed.

Conversely, suppose A is closed. $\Rightarrow$ the smallest closed set containing A is A itself.

$\therefore A = \bar{A}$.

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Note: i) $\Phi = \overline{\Phi}$ ii) $M = \overline{M}$ iii) $\bar{A} = \bar{\bar{A}}$

Ex 1: Consider $\mathbb{R}$ with usual metric.

a) Let $A = [0,1]$.

W.K. that A is closed set.

$\therefore \bar{A} = A = [0,1]$

b) Let $A = (0,1)$. $\Rightarrow [0,1]$ is closed set containing $(0,1)$.

Obviously $[0,1]$ is the smallest closed set containing $(0,1)$.

$\therefore \bar{A} = [0,1]$.

Ex 2: In a discrete metric space (M, d) any subset A of M is closed.

Soln: Let (M, d) be a discrete metric space.

Let $A \subseteq M$.

$\because$ Every subset of a discrete metric space is open A^c is open.

$\therefore A$ is closed.

Hence $\bar{A} = A$.

Theorem 2.14: Let (M, d) be a metric space. Let $A, B \subseteq M$.

Then i) $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$

ii) $\overline{A \cup B} = \bar{A} \cup \bar{B}$

iii) $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

Proof:

i) Let $A \subseteq B$.

Now $\bar{B} \supseteq B \supseteq A$

$\therefore \bar{B}$ is closed set containing A.

But $\bar{A}$ is the smallest closed set containing A.

$\therefore \bar{A} \subseteq \bar{B}$

ii) We have $A \subseteq A \cup B$

$\therefore \bar{A} \subseteq \overline{A \cup B}$ (by ①)

Similarly $\bar{B} \subseteq \overline{A \cup B}$

$\therefore \bar{A} \cup \bar{B} \subseteq \overline{A \cup B} \longrightarrow$ ①

Now, $\bar{A}$ is a closed set containing A and $\bar{B}$ is a closed set containing B.

$\therefore \bar{A} \cup \bar{B}$ is a closed set containing $A \cup B$.

$\overline{A \cup B}$ is the smallest closed set containing $A \cup B$.

$\therefore \overline{A \cup B} \subseteq \bar{A} \cup \bar{B} \longrightarrow$ ②

from ① & ② we get

$$\overline{A \cup B} = \bar{A} \cup \bar{B}$$

iii) We have $A \cap B \subseteq A$

$\overline{A \cap B} \subseteq \bar{A}$ (by i)

Similarly $\overline{A \cap B} \subseteq \bar{B}$

$$\therefore \overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$$

Note: $\overline{A \cap B}$ need not be equal to $\bar{A} \cap \bar{B}$

Ex; $\mathbb{R}$ with usual metric,

$A = (0,1)$ & $B = (1,2)$

$\Rightarrow A \cap B = \phi$

$\therefore \overline{A \cap B} = \bar{\phi}$

But $\bar{A} \cap \bar{B} = [0,1] \cap [1,2]$

$= \{1\}$

$\therefore \overline{A \cap B} \neq \bar{A} \cap \bar{B}$

Limit Point:

Definition: Let (M,d) be a metric space. Let $A \subseteq M$. Let $x \in M$. Then x is called a limit point or a cluster point or an accumulation point of A if every open ball with centre x contains at least one point of A different from x.

(i.e.), $B(x,r) \cap (A - \{x\}) \neq \phi \quad \forall\, r > 0$.

The set of all limit points of A is called the derived set of A and denoted by $D(A)$.

<u>Note</u>: x is not a limit point of A iff there exists an open ball $B(x,r)$ such that $B(x,r) \cap (A - \{x\}) = \phi$.

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consider $\mathbb{R}$ with usual metric.

a) Let $A = [0,1)$.

Any open ball with centre 0 is of the form $(-r,r)$ which contains a point of $[0,1)$ other than 0.

$\therefore$ 0 is a limit point of $[0,1)$

Similarly, 1 is a limit point of $[0,1)$

2 is not a limit point of A, since

$$\left(2-\tfrac{1}{2},\ 2+\tfrac{1}{2}\right) \cap [0,1) = \left(\tfrac{3}{2},\tfrac{5}{2}\right) \cap [0,1) = \phi$$

all points of $[0,1)$ are limit points of $[0,1)$ & no other point is a limit point.

Hence $D[0,1) = [0,1]$.

b) Let $A = \{1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{n}, \dots\}$.

Here 0 is a limit point of A.

Consider any open ball $(-r,r)$ with centre 0.

Choose a positive integer n such that $\frac{1}{n} < r$.

$\Rightarrow \frac{1}{n} \in (-r, r)$.

$\therefore (-r,r)$ contains a point of A which is different from 0.

$\therefore$ 0 is a limit point of A.

1 is not a limit point of A.

$$\left(1-\tfrac{1}{4},\ 1+\tfrac{1}{4}\right) \cap (A-\{1\}) = \left(\tfrac{3}{4},\tfrac{5}{4}\right) \cap \left\{\tfrac{1}{2},\tfrac{1}{3},\dots\tfrac{1}{n},\dots\right\} = \phi$$

If any point except zero is not a limit point of A

Ex: If take a 2

$(2-\frac{1}{4}, 2+\frac{1}{4}) \cap (A-\{2\}) = (\frac{7}{4}, \frac{9}{4}) \cap \{-1, \dots\} = \phi$

$\therefore$ 2 is not a limit point

$\therefore$ $D(A) = \{0\}$.

c) Z has no limit point.

<u>Proof</u>: Let x be any real number.

<u>If x is an integer</u>, $\Rightarrow B(x, \frac{1}{2}) = (x-\frac{1}{2}, x+\frac{1}{2})$ does not contain any integer other than x.

$\therefore$ x is not a limit point of Z.

<u>If x is not an integer</u>, let n be the integer which is closest to x.

Choose r such that $0 < r < |x-n|$.

$\Rightarrow B(x, r) = (x-r, x+r)$ contains no integer.

$\therefore$ x is not a limit point of Z.

Since x is arbitrary Z has no limit point.

$\therefore$ $D(Z) = \phi$.

d) consider Q.

Any real number x is a limit point of Q, Since any interval $(x-r, x+r)$ contains infinite number of rational numbers.

$\therefore$ $D(Q) = R$.

<u>Ex 3</u>: Let (M, d) be a discrete metric space.

(7)

Let $A \subseteq M$.

Let $x \in M$.

$\Rightarrow B(x, \frac{1}{2}) \cap (A-\{x\}) = \{x\} \cap (A-\{x\}) = \phi$.

$\therefore$ x is not a limit point of A.

Since $x \in M$ is arbitrary A has no limit point.

$\therefore$ $D(A) = \phi$

Thus any subset of a discrete metric space has no limit point.

Ex: 4 consider $\mathbb{C}$ with usual metric.

Let $A = \{z \mid |z| < 1\}$

$\Rightarrow D(A) = \{z \mid |z| \leq 1\}$.

Theorem 2.15: Let (M, d) be a metric space. Let $A \subseteq M$. Then x is a limit point of A iff each open ball with centre x contains an infinite number of points of A.

(7)

Proof: Let x be a limit point of A.

Suppose an open ball $B(x, r)$ contains only a finite number of points of A.

Let $B(x, r) \cap (A - \{x\}) = \{x_1, x_2, \dots x_n\}$.

Let $r_1 = \min\{d(x, x_i) \mid i = 1, 2, \dots n\}$.

Since $x \neq x_i$, $d(x, x_i) > 0 \quad \forall i = 1, 2, \dots, n$

and hence $r_1 > 0$.

Also $B(x, r_1) \cap (A - \{x\}) = \phi$.

$\therefore x$ is not a limit point of A which is a contradiction.

$\therefore$ Every open ball with centre x contains infinite number of points of A.

$\therefore$ The converse is obvious.

Corollary: Any finite subset of a metric space has no limit point.

Proof: Let A be a finite subset of M.

Suppose A has limit point say x. Then $B(x, r)$ contains infinite number of points of A.

This is a contradiction since A is finite.

Theorem 2.16 : Let M be a metric space and $A \subseteq M$. Then $\bar{A} = A \cup D(A)$.

Proof :

Let $x \in A \cup D(A)$.

To prove : $x \in \bar{A}$.

Suppose $x \notin \bar{A}$

$\therefore x \in M - \bar{A}$ & since $\bar{A}$ is closed $M - \bar{A}$ is open

$\therefore$ There exists an open ball $B(x, r) \subseteq M - \bar{A}$.

$\therefore B(x, r) \cap \bar{A} = \phi$

$\therefore B(x, r) \cap A = \phi$ $\quad (\because A \subseteq \bar{A})$

$x \notin A \cup D(A)$ which is a contradiction.

Hence proved $x \in \bar{A}$.

$\therefore A \cup D(A) \subseteq \bar{A} \longrightarrow (1)$

Now let $x \in \bar{A}$.

To prove : $x \in A \cup D(A)$.

If $x \in A$ clearly $x \in A \cup D(A)$

Suppose $x \notin A$.

To prove : $x \in$ ~~A∪~~ $D(A)$

Suppose $x \notin D(A)$.

$\Rightarrow$ there exists an open ball $B(x, r)$ such that $B(x, r) \cap A = \phi$.

$\therefore B(x, r)^c \supseteq A$ & $B(x, r)^c$ is closed.

But $\bar{A}$ is the smallest closed set containing A.

$\therefore \bar{A} \subseteq B(x, r)^c$.

But $x \in \bar{A}$ & $x \notin B(x, r)^c$ which is contradiction

Hence $x \in D(A)$

$\therefore x \in A \cup D(A)$

$\therefore \bar{A} \subseteq A \cup D(A)$. $\longrightarrow (2)$

from (1) & (2) we get $\bar{A} = A \cup D(A)$.

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Corollary 1: A is closed iff A contains all its limit points.

i.e) A is closed iff $D(A) \subseteq A$.

Proof: A is closed $\Leftrightarrow A = \bar{A}$ (since $\bar{A}$ is closed A is closed)

$\Leftrightarrow A = A \cup D(A)$

$\Leftrightarrow D(A) \subseteq A$.

Corollary 2:

$$x \in \bar{A} \Leftrightarrow B(x,r) \cap A \neq \phi \text{ for all } r > 0.$$

Proof:

Let $x \in \bar{A}$, Then $x \in A \cup D(A)$.

$\therefore x \in A$ or $x \in D(A)$,

If $x \in A \Rightarrow x \in B(x,r) \cap A$.

If $x \in D(A) \Rightarrow B(x,r) \cap A \neq \phi \quad \forall r > 0$.

$\therefore$ in both cases $B(x,r) \cap A \neq \phi \quad \forall r > 0$.

conversely, suppose $B(x,r) \cap A \neq \phi \quad \forall r > 0$.

To prove: $x \in \bar{A}$.

If $x \in A$ trivially $x \in \bar{A}$

Let $x \notin A$. $\Rightarrow A - \{x\} = A$,

$\therefore B(x,r) \cap (A - \{x\}) \neq \phi$

$\therefore x \in D(A)$

$\therefore x \in \bar{A}$

Corollary 3: $x \in \bar{A} \Leftrightarrow G \cap A \neq \phi$ for every open set G containing x.

Proof: Let $x \in \bar{A}$. Let G be an open set containing x. $\Rightarrow$ there exists $r > 0$ such that $B(x,r) \subseteq G$,

Also, $\because x \in \bar{A}$, $B(x,r) \cap A \neq \phi$

$\therefore G \cap A \neq \phi$,

Conversely, suppose $G \cap A \neq \phi$ for every open set G containing x.

since $B(x,r)$ is an open set containing x, we have $B(x,r) \cap A \neq \phi$. $\therefore x \in \bar{A}$.

Ex 1: Consider R with usual metric.

a) Let $A = [0, 1)$

$$\Rightarrow \bar{A} = A \cup D(A)$$
$$= [0,1) \cup [0,1]$$
$$= [0,1].$$

b) Let $A = \{1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\}$

$$\Rightarrow \bar{A} = A \cup D(A)$$
$$= \{1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\} \cup \{0\}.$$

c) $\bar{Z} = Z \cup D(Z)$

$$= Z \cup \phi = Z.$$

$\therefore Z$ is closed.

d) $\bar{Q} = Q \cup D(Q)$

$$= Q \cup R.$$
$$= R.$$

$\therefore Q$ is not closed.

Ex 2: In $R \times R$ with usual metric.

$$\overline{Q \times Q} = (Q \times Q) \cup D(Q \times Q)$$
$$= (Q \times Q) \cup (R \times R)$$
$$= R \times R.$$

$\therefore (Q \times Q)$ is not closed

Solved Problem:

Pbm 1: Prove that for any subset A of a metric space, $d(A) = d(\bar{A})$ where $d(A)$ is the diameter of A.

Soln.

we have $A \subseteq \bar{A}$.

$$\therefore d(A) \leq d(\bar{A}) \longrightarrow (1)$$

Now, let $\epsilon > 0$ be given

To prove:

$$d(\bar{A}) \le d(A) + \varepsilon.$$

Let $x, y \in \bar{A}$.

$\therefore B(x, \frac{1}{2}\varepsilon) \cap A \neq \phi$

$B(y, \frac{1}{2}\varepsilon) \cap A \neq \phi$

Let $x_1 \in (x, \frac{1}{2}\varepsilon) \cap A$

$x_2 \in (y, \frac{1}{2}\varepsilon) \cap A$.

$\therefore x_1 \in (x, \frac{1}{2}\varepsilon)$ & $x_2 \in (y, \frac{1}{2}\varepsilon)$

$\therefore$ $\left.\begin{array}{l} d(x, x_1) < \frac{1}{2}\varepsilon \text{ \&} \\ d(y, y_2) < \frac{1}{2}\varepsilon \end{array}\right\} \longrightarrow (2)$

Also $x_1 \in A$ & $x_2 \in A$

$\Rightarrow d(x_1, x_2) \le d(A) \rightarrow (3)$

Now,

$d(x, y) \le d(x, x_1) + d(x_1, x_2) + d(x_2, y)$.

$< \frac{1}{2}\varepsilon + d(A) + \frac{1}{2}\varepsilon$ (by (2) & (3))

$= d(A) + \varepsilon$.

Thus $d(x, y) < d(A) + \varepsilon$.

dA = least upper bound $\{d(x, y) / x, y \in \bar{A}\} \le d(A) + \varepsilon$

i.e) $d(\bar{A}) \le d(A) + \varepsilon$

Since ε is arbitrary, we have $d(\bar{A}) \le d(A)$ $\rightarrow (4)$

By (1) & (4) we get,

$d(A) = d(\bar{A})$

Dense Sets:

Definition: A subset A of a metric space M is said to be dense in M or every where dense if $\bar{A} = M$.

Definition: A metric space M is said to be seperable if there exists a countable dense subset in M.